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# Lattices and Quantum Logics with Separated Intervals, Atomicity

**Zdenka Riečanová**[1]

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It is well known that a Boolean algebra $B$ is atomic (atomistic) iff the interval topology on $B$ is Hausdorff. But this no longer holds for orthomodular lattices (quantum logics). There exist (even complete) atomic orthomodular lattices the interval topology of which is not Hausdorff. We show that another characterization of atomicity for Boolean algebras is the following: A Boolean algebra $B$ is atomic iff $B$ has separated intervals. Furthermore, we show that the interval topology on a complete orthomodular lattice $L$ is Hausdorff iff $L$ has separated intervals iff $L$ is atomic and it has separated intervals. An orthomodular lattice $L$ with orthomodular MacNeille completion $\hat{L}$ has separated intervals iff $L$ is atomic and it has separated intervals iff the interval topology on $\hat{L}$ is Hausdorff.

## 1. PRELIMINARIES

Recall that the *interval topology* $\tau_i$ on a poset $P$ is the smallest topology in which all *closed intervals*

$$[a) = \{x \in P | a \leq x\}, \qquad (a] = \{x \in P | x \leq a\}$$

$$[a, a] = \{a\}, \qquad [a, b] = \{x \in P | a \leq x \leq b\}, \qquad \text{and} \quad P$$

are closed sets. A base for the closed sets is then the collection of all finite unions of such intervals. Thus the set complements of finite unions of closed intervals generate the open base of $\tau_i$. Evidently, $\tau_i$ on any poset $P$ is $T_1$ (each singleton is closed in $\tau_i$).

By theorem of Frink (1942) the interval topology $\tau_i$ on a lattice is compact iff $L$ is complete. Observing that a compact Hausdorff topology is always normal, we infer that the separation axioms Hausdorff, regular,

[1] Department of Mathematics, Faculty of Electrical Engineering and Information Technology, Slovak Technical University, 812 19 Bratislava, Slovak Republic; e-mail: zriecan@elf.stuba.sk.



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completely regular, and normal are equivalent for the interval topology $\tau_i$ on every complete lattice.

Every poset $P$ can be embedded into a complete lattice. A complete lattice $\hat{P}$ into which $P$ is embedded is called a MacNeille completion of $P$ if $P$ is supremum and infimum dense in $\hat{P}$ (i.e., for every $x \in \hat{P}$ there are $M, Q \subseteq P$ with $\vee M = \wedge Q = x;$ we identify here $P$ with $\varphi(P)$, where $\varphi$: $P \to \hat{P}$ is the embedding). Then the embedding preserves all suprema and infima existing in $P$ (Schmidt, 1956).

Henceforth, we denote $[a, b]_P = \{x \in P | a \leq x \leq b\}$ and $[c, d]_{\hat{P}} = \{x \in \hat{P} | c \leq x \leq d\}$ for $a, b \in P$ and $c, d \in \hat{P}$. Similarly for other closed intervals.

*Lemma 1.1.* For the interval topology $\tau_i$ on a poset $P$ and the interval topology $\hat{\tau}_i$ on the MacNeille completion $\hat{P}$ of $P$, the following conditions are satisfied:

(i) $\hat{\tau}_i \cap P = \tau_i$.

(ii) $\hat{\tau}_i$ is $T_2 \Rightarrow \tau_i$ is $T_2$.

(iii) $\tau_i$ is $T_2$ does not imply $\hat{\tau}_i$ is $T_2$.

(iv) If $P$ is a lattice, then $[a, b]_P \cap [c, d]_P = \emptyset$ iff $[a, b]_{\hat{P}} \cap [c, d]_{\hat{P}} = \emptyset$; $a, b, c, d \in P$ (similarly for all closed intervals).

*Proof.* (i) This is obvious. Observing only that for $a, b \in \hat{P}$ we have $[a, b]_{\hat{P}} \cap P = \bigcap_{x \leq a \leq b \leq y, x, y \in P} [x, y]_P$. Similarly for other closed intervals.

(ii) It follows trivially from (a).

(iii) Suppose $P$ is a poset with Hausdorff, not regular, interval topology $\tau_i$ [for an example of such $P$ see Erné (1980)]. Assume that $\hat{\tau}_i$ on $\hat{P}$ is $T_2$. Then $\hat{\tau}_i$ is regular and by (a), $\tau_i = \hat{\tau}_i \cap P$ is also regular, a contradiction. We conclude $\hat{\tau}_i$ is not $T_2$.

(iv) Suppose $x \in [a, b]_{\hat{P}} \cap [c, d]_{\hat{P}}$. Then conditions $a \vee c \leq x \leq b \wedge d$ imply $a \vee c$, $b \wedge d \in [a, b]_P \cap [c, d]_P$.

We say that a net $(x_\alpha)_{\alpha \in \mathscr{E}}$ of elements of $P$ ($\mathscr{E}$ is a directed set) *order converges* to a point $x \in P$ if there exist nets $(u_\alpha)_{\alpha \in \mathscr{E}}, (v_\alpha)_{\alpha \in \mathscr{E}} \subseteq P$ such that $u_\alpha \leq x_\alpha \leq v_\alpha$ for all $\alpha$ and $(u_\alpha)_{\alpha \in \mathscr{E}}$ is nondecreasing with supremum $x$, $(v_\alpha)_{\alpha \in \mathscr{E}}$ is noincreasing with infimum $x$. We write $u_\alpha \uparrow x$, $v_\alpha \downarrow x$, and $x_\alpha \xrightarrow{(o)} x$. The finest (biggest) topology on $P$ such that $x_\alpha \xrightarrow{(o)} x$ implies topological convergence is called an *order topology on P*, denoted $\tau_o$. On every lattice, $\tau_i \subseteq \tau_o$ [i.e, $x_\alpha \xrightarrow{(o)} x$ implies $x_\alpha \xrightarrow{\tau_i} x$ and if $\tau_i$ is $T_2$, then $\tau_i = \tau_o$ (e.g., Erné and Riečanová, 1995)]. Thus, if $L$ is a lattice, then for every element $x$ of its MacNeille completion $\hat{L}$ there is a net $(x_\alpha)_\alpha \subseteq L$ with $x_\alpha \xrightarrow{\hat{\tau}_i} x$. Actually, if $x \in \hat{L}$, then there exists either $\emptyset \neq M \subseteq L$ with $\vee M = x$, or $\emptyset \neq Q \subseteq L$ with $\wedge Q = x$. We put $x_\alpha = \vee \alpha$ for all finite $\alpha \subseteq M$, or $x_\alpha = \wedge \alpha$ for all finite $\alpha \subseteq Q$. Then $x_\alpha \uparrow x$ or $x_\alpha \downarrow x$. In both cases $x_\alpha \xrightarrow{(o)} x$ (in $\hat{L}$), which implies $x_\alpha \xrightarrow{\tau_i} x$.

## 2. BOUNDED LATTICES WITH SEPARATED INTERVALS

*Definition 2.1.* We say that a *bounded lattice L has separated intervals* if given any two disjoint intervals $[a, b]$, $[c, d] \subseteq L$, the lattice $L$ can be covered by a finite number of closed intervals each of which is disjoint with at least one of the intervals $[a, b]$ and $[c, d]$.

If a lattice $L$ is bounded, then we denote by 0 the smallest element and by 1 the largest element of $L$. It should be noted that if $L = \cup_{k=1}^{u} [a_k\ b_k]_L$, then the MacNeille completion $\hat{L} = \cup_{k=1}^{u} [a_k, b_k]_{\hat{L}}$. Actually, if $x \in \hat{L} \backslash \cup_{k=1}^{u} [a_k, b_k]_{\hat{L}} \in \hat{\tau}_1$, then there exists $(x_\alpha)_\alpha \subseteq L$ with $x_\alpha \xrightarrow{\hat{\tau}_i} x$, which implies that there exists $\alpha_0$ such that $x_\alpha \in \hat{L} \backslash \cup\ [a_k, b_k]_{\hat{L}} \subseteq \hat{L} \backslash L$ for all $\alpha \geq \alpha_0$, a contradiction.

*Lemma 2.2.* The interval topology $\tau_i$ on a complete lattice $L$ is Hausdorff iff $L$ has separated intervals.

*Proof.* (1) Suppose that $\tau_i$ is Hausdorff and $[a, b]$ $[c, d] \subseteq L$ are disjoint. Since $\tau_i$ is compact and normal, there are disjoint sets from the open base of $\tau_i$

$$\mathcal{U} = L \backslash \bigcup_{k=1}^{n} [u_k, v_k], \qquad \mathcal{V} = L \backslash \bigcup_{l=1}^{m} [w_l, z_l]$$

such that $[a, b] \subseteq \mathcal{U}$ and $[c, d] \subseteq \mathcal{V}$. Thus $L = (\cup_{k=1}^{n} [u_k, v_k]) \cup (\cup_{l=1}^{m} [w_l, z_l])$ and each of the intervals $[u_k, v_k]$, $[w_l, z_l]$ is disjoint with at least one of $[a, b]$ and $[c, d]$.

(2) Since for $x, y \in L$ with $x \neq y$ the singletons $\{x\}$, $\{y\}$ are disjoint closed intervals in $L$, the condition $L$ has separated intervals implies $\tau_i$ is Hausdorff.

*Theorem 2.3.* Let a bounded lattice $L$ have separated intervals. Then the interval topology $\hat{\tau}_i$ on the MacNeille completion $\hat{L}$ of $L$ is Hausdorff (normal).

*Proof.* Let $x \neq y$ and $x, y \in \hat{L}\ \{0, 1\}$. Then there exists $u \in L$ with $u \leq x$ and $u \not\leq y$ (or $u \leq y$ and $u \not\leq x$). Hence $x \in [u, 1]_{\hat{L}}$ and $y \in \hat{L} \backslash [u, 1]_{\hat{L}} \in \hat{\tau}_i$. Let $M, Q \subseteq L$ with $\vee M = \wedge Q = y$. Let the set $\varepsilon = \{\alpha \subseteq M \cup Q | \alpha$ is finite and $\alpha \cap M \neq \emptyset \neq \alpha \cap Q\}$ be directed by set inclusion. Denote $y_\alpha = \vee \alpha \cap M$ and $z_\alpha = \wedge \alpha \cap Q$ for all $\alpha \in \varepsilon$. Then $y_\alpha \uparrow y$ and $z_\alpha \downarrow y$. Suppose that for all $\alpha \in \varepsilon$ there exist $u_\alpha \notin \hat{L} \backslash [u, 1]_{\hat{L}}$ such that $y_\alpha \leq u_\alpha \leq z_\alpha$. Then $u_\alpha \xrightarrow{(o)} y$ (in $\hat{L}$) and hence $u_\alpha \xrightarrow{\tau_i} y$, a contradiction. We conclude that there exists $a_0 \in \varepsilon$ such that $[y_{\alpha_0}, z_{\alpha_0}]_{\hat{L}} \subseteq \hat{L} \backslash [u, 1]_{\hat{L}}$. It follows that $[y_{\alpha_0}, z_{\alpha_0}]_{\hat{L}} \cap [u, 1]_{\hat{L}} = \emptyset$. By the assumption there exist $u_k, v_k, w_l, z_l \in L$

such that $L = (\cup_{k=1}^{n} [u_k, v_k]_L) \cup (\cup_{l=1}^{m} [w_l, z_l]_L)$ and $[u_k, v_k]_L \cap [y_{\alpha_0}, z_{\alpha_0}] = \emptyset$, $[w_l, z_l]_L \cap [u, 1]_L = \emptyset$, for $k = 1, \ldots, n$; $l = 1, \ldots, m$. It follows that $\hat{L} = (\cup_{k=1}^{n} [u_k, v_k]_{\hat{L}}) \cup (\cup_{l=1}^{m} [w_l, z_l]_{\hat{L}})$ and each of the intervals $[u_k, v_k]_{\hat{L}}$, $[w_1, z_l]_{\hat{L}}$ ($k = 1, \ldots, n$; $l = 1 \ldots, m$) includes at most one of the points $x$ and $y$.

Suppose now $x = 0$ and $y \neq 0$. Then there exists $v \in L$ with $0 \neq v < y$. Moreover, $0 \neq v$ implies that there exists $u \in L$ with $v \not\leq u$. We conclude that $[0, u]_L \cap [v, 1]_L = \emptyset$ and $y \in [v, 1]_{\hat{L}}$. Now, by the assumption, $L = \cup_{k=1}^{n} [a_k, b_k]_L$ and each of the intervals $[a_k, b_k]_L$ is disjoint with at least one of $[0, u]_L$ and $[v, 1]_L$. It follows that $\hat{L} = \cup_{k=1}^{u} [a_k, b_k]_{\hat{L}}$ and each of the intervals $[a_k, b_k]_{\hat{L}}$, includes at most one of the points $x$ and $y$. Similarly if $y = 0$ and $x \neq 0$.

## 3. QUANTUM LOGICS WITH SEPARATED INTERVALS ATOMICITY

*Definition 3.1* An *ortholattice* is a lattice $L$ with a least element 0 and a greatest element 1 and with a unary operation $'$ called orthocomplementation such that for all $a, b \in L$

(i) $(a')' = a$
(ii) $a \leq b$ implies $b' \leq a'$
(iii) $a \wedge a' = 1$

A pair $a, b \in L$ is called orthogonal if $a \leq b'$. An ortholattice in which for all $a, b \in L$, if $a \leq b$, then $b = a \vee (a' \wedge b)$ (the orthomodular law), is called an *orthomodular lattice* (*quantum logic*).

An orthomodular lattice is a *Boolean algebra* iff it is a distributive lattice. A maximal Boolean subalgebra of an orthomodular lattice $L$ is called a *block*.

An ortholattice $L$ is *atomic* if every nonzero element of $L$ is over an atom and $L$ is *atomistic* if every element of $L$ is the supremum of all atoms under it. An orthomodular lattice $L$ is atomic iff $L$ is atomistic. On the other hand, there are atomic ortholattices which are not atomistic (Kalmbach, 1983).

It is known that the MacNeille completion of an orthomodular lattice need not be orthomodular. On the other hand, there are some positive results on orthomodular MacNeille completions (e.g., Harding, 1993; Riečanová, 1994).

*Theorem 3.2.* (Riečanová, 1994). For an ortholattice $L$ the following conditions are equivalent.

(i) $L$ and the MacNeille completion $\hat{L}$ of $L$ are orthomodular lattices.

(ii) For all $M_1 \subseteq M_2 \subseteq L$ there are orthogonal sets $Q_1 \subseteq Q_2 \subseteq L$ (i.e., $x \leq y^{\perp}$ for all $x \neq y$, $x, y \in Q_2$) such that $\underline{M_1} = \underline{Q_1}$ and $\underline{M_2} = \underline{Q_2}$. Here, for any $M \subseteq L$, we put $\underline{M} = \{y \in L | x \leq y \text{ for all } x \in M\}$.

Note that for $M, Q \subseteq L$ we have $\overline{M} = \overline{Q}$ iff $\vee M = \vee Q$ in the MacNeille completion $\hat{L}$ for $L$.

*Theorem 3.3.* For an atomistic ortholattice. $L$, the following conditions are equivalent:

(i) The interval topology $\hat{\tau}_i$ on the MacNeille completion $\hat{L}$ of $L$ is Hausdorff.

(ii) $L$ has separated intervals.

(iii) For every pair of disjoint closed intervals $[a, b]_L, [c, d]_L \subseteq L$ there are atoms $p_k, q_l \in L$ ($k = 1, \dots, n;\ l = 1 \dots, m$), such that $L = (\cup_{k=1}^{n} [p_k, 1]_L) \cup (\cup_{l=1}^{m} [0, q_l^{\perp}]_L)$ and each of the intervals $[p_k, 1]_L, [0, q_l^{\perp}]_L$ is disjoint with at least one of $[a, b]_L$ and $[c, d]_L$.

*Proof.* (i) $\Rightarrow$ (iii): Suppose that $[a, b]_L$ and $[c, d]_L$ are disjoint intervals in $L$. Then $[a, b]_{\hat{L}} \cap [c, d]_{\hat{L}} = \emptyset$. By Lemma 2.2 there exist $u_k, v_k, w_l, z_l \in \hat{L}$ such that $\hat{L} = (\cup_{k=1}^{n} [u_k, v_k]_{\hat{L}}) \cup (\cup_{l=1}^{m} [w_l, z_l]_{\hat{L}})$ and $[u_k, v_k]_{\hat{L}} \cap [c, d]_{\hat{L}} = \emptyset$ for $k = 1, \dots, n$, as well as $[w_l, z_l]_{\hat{L}} \cap [a, b]_{\hat{L}} = \emptyset$ for $\mathscr{L} = 1, \dots, m$.

For $k = 1, 2, \dots, n$, the condition $[u_k, u_k]_{\hat{L}} \cap [c, d]_{\hat{L}} = \emptyset$ is equivalent to $u_k \vee c \not\leq v_k \wedge d$, which is equivalent to: either $u_k \not\leq v_k \wedge d$ or $c \not\leq v_k \wedge d$. The condition $u_k \not\leq v_k \wedge d$ is equivalent to the existence of an atom $a_k \in L$ such that $a_k \leq u_k$ and $a_k \not\leq d$. Thus $a_k \leq u_k \leq v_k \leq 1$ and $[a_k, 1]_{\hat{L}} \cap [c, d]_{\hat{L}} = \emptyset$. Then we can consider $[a_k, 1]_{\hat{L}}$ instead of $[u_k, v_k]_L$. The condition $c \not\leq v_k \wedge d$ is equivalent to the existence of an atom $b_k \in L$ with $b_k^{\perp} \geq v_k$ and $b_k^{\perp} \not\geq c\}$. Then $0 \leq u_k \leq v_k \leq b_k^{\perp}$ and $[0, b_k^{\perp}]_{\hat{L}} \cap [c, d]_{\hat{L}} = \emptyset$. In this case we can consider $[0, b_k^{\perp}]_{\hat{L}}$ instead of $[u_k, v_k]_{\hat{L}}$.

Similarly we can consider $[c_l, 1]_{\hat{L}}$ or $[0, d_l^{\perp}])_{\hat{L}}$ instead of $[w_l, z_l]_{\hat{L}}$, for $l = 1, \dots, m$ and some atoms $c_l, d_l \in L$.

Since for any atoms $a, b \in L$ we have $[a, 1]_{\hat{L}} \cap L = [a, 1]_{\hat{L}}$ and $[0, b^{\perp}]_{\hat{L}} \cap L = [0, b]_L$, we can conclude that (iii) is satisfied.

(iii) $\Rightarrow$ (ii) is obvious.

(ii) $\Rightarrow$ (i) follows by Theorem 2.3.

Since the MacNeille completion $\hat{B}$ of any Boolean algebra $B$ is a complete Boolean algebra and $\hat{B}$ is atomic iff $B$ is atomic, we have that $\hat{\tau}_i$ on $\hat{B}$ is $T_2$ iff $\tau_i$ on $B$ is $T_2$. We obtain the following corollary:

*Corollary 3.4.* A Boolean algebra $B$ is atomic iff $B$ has separated intervals iff $\tau_i$ on $B$ is Hausdorff.

An atom of a block of an orthomodular lattice $L$ is also an atom of $L$. On the other hand, if $L$ is an atomic orthomodular lattice, then, in general, every block in $L$ need not be atomic. For example, on the complete atomic orthomodular lattice $L(H)$ of all closed linear subspaces of a complex separable infinite-dimensional Hilbert space $H$ the range of the spectral measure corresponding to a self-adjoint operator with a simple-continuous spectrum (e.g., the 'position' or 'momentum' operator) is an atomless block of $L(H)$ (Beltrametti and Cassinclli, 1981, pp. 21, 38).

*Theorem 3.5.* For an orthomodular lattice $L$ with orthomodular MacNeille completion $\hat{L}$, the following conditions are equivalent:

(a) $\tau_i$ on $L$ and $\hat{\tau}_i$ on $\hat{L}$ are Hausdorff.

(b) $L$ has separated intervals.

(c) $L$ is atomic with separated intervals.

(d) For every pair of disjoint closed intervals $[a, b]_L$, $[c, d]_{\hat{L}} \subseteq L$ there are atoms $p_k, q_l \in L$ ($k = 1, \ldots, n$; $l = 1, \ldots, m$) such that $L = (\cup_{k=1}^{n} [p_k, 1]_L) \cup (\cup_{l=1}^{m} [0, q_l^{\perp}]_{\hat{L}})$ and each of the intervals $[p_k, 1]_L$, $[0, q_l^{\perp}]_{\hat{L}}$ is disjoint with at least one of $[a, b]_L$ and $[c, d]_L$.

If one of the conditions $(a)$–$(d)$ holds, then all blocks in $L$ and $\hat{L}$ are atomic. Moreover, every block $\hat{B}$ of $\hat{L}$ is the MacNeille completion of the block $B = \hat{B} \cap L$; $\hat{B}$ is isomorphic to the power set of some maximal orthogonal set of atoms of $L$ and it is a MacNeille completion of a block in $L$.

*Proof.* The condition (a) implies that every block in $\hat{L}$ is atomic. This is because for the interval topology $\tau_i^{\hat{B}}$ on $\hat{B}$ we have $\tau_i^{\hat{B}} = \tau_i^{\hat{B}} \cap \hat{B}$ (*since* $\hat{B}$ is subcomplete in $\hat{L}$), hence $\tau_i^{\hat{B}}$ is $T_2$. Moreover, every atom of $\hat{B}$ is an atom of $\hat{L}$ and hence an atom of $L$. Clearly, the set of all atoms of $\hat{B}$ is a maximal orthogonal set of atoms of $L$ (i.e., the set of all atoms of a block $B = \hat{B} \cap L$ of $L$). Since $\hat{B}$ is complete and every $x \in \hat{B}$ is a supremum of atoms of $B$, we conclude that $\hat{B}$ is the MacNeille completion of the block $B = \hat{B} \cap L$.

If $\hat{\tau}_i$ on $\hat{L}$ is $T_2$, then $\hat{L}$ is atomic and hence atomistic. Thus (a) $\Rightarrow$ (b), (c), and (d) by Theorem 3.3. It is obvious that (c) $\Rightarrow$ (b) and (d) $\Rightarrow$ (b). By Theorem 2.3, (b) $\Rightarrow$ (a).

*Corollary 3.6.* For a complete orthomodular lattice $L$ the following conditions are equivalent:

(a) $\tau_i$ on $L$ is Hausdorff.

(b) $L$ has separated intervals.

(c) $L$ is atomic with separated intervals.

(d) For every pair of disjoint closed intervals $[a, b]_L$, $[c, d]_L \subseteq L$ there are atoms $p_k, q_l \in L$ ($k = 1, \ldots, n$; $l = 1, \ldots, m$) such that $L = (\cup_{k=1}^{n}, [p_k, 1]_L) \cup (\cup_{l=1}^{m} [0, g_l^{\perp}]_L)$ and each of the intervals $[p_k, 1]_L$, $[0, q_l^{\perp}]_L$ is disjoint with at least one of $[a, b]_L$ and $[c, d]_L$.

*Proof.* The proof is straightforward. Observing only that if $\tau_i$ on $L$ is Hausdorff, then $L$ is atomic, since every block of $L$ is atomic.

Note that if all blocks of a (complete) orthomodular lattice $L$ are atomic, then $\tau_i$ on $L$ need not be Hausdorff; for example, $L(H)$, where $H$ is a finitedimensional Hilbert space with dim $H \geq 2$. Evidently, $\tau_i$ on $L(H)$ is not $T_2$ [since $L(H)$ does not have separated intervals], but every block of $L(H)$ is atomic.

*Example 3.7.* Examples of orthomodular lattices which satisfy conditions of Corollary 3.6 are compact order-topological orthomodular lattices. Almost orthogonal atomic orthomodular lattices satisfy conditions of Theorem 3.5 [see Erné and Riečanová (1995) for further references].

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